Algorithmic Information Theory and Foundations of Probability

Alexander Shen*

Abstract

The question how and why mathematical probability theory can be applied to the "real world" has been debated for centuries. We try to survey the role of algorithmic information theory (Kolmogorov complexity) in this debate.

1 Probability theory paradox

One often describes the natural sciences framework as follows: a hypothesis is used to predict something, and the prediction is then checked against the observed actual behavior of the system; if there is a contradiction, the hypothesis needs to be changed.

Can we include probability theory in this framework? A statistical hypothesis (say, the assumption of a fair coin) should be then checked against the experimental data (results of coin tossing) and rejected if some discrepancy is found. However, there is an obvious problem: The fair coin assumption says that in a series of, say, 1000 coin tossings all the 2^{1000} possible outcomes (all 2^{1000} bit strings of length 1000) have the same probability 2^{-1000}. How can we say that some of them contradict the assumption while other do not?

The same paradox can be explained in a different way. Consider a casino that wants to outsource the task of card shuffling to a special factory that produced shrink-wrapped well shuffled decks of cards. This factory would need some quality control department. It looks at the deck before shipping it to the customer, blocks some "badly shuffled" decks and approves some others as "well shuffled". But how is it possible if all $n!$ orderings of n cards have the same probability?

2 Current best practice

Whatever the philosophers say, statisticians have to perform their duties. Let us try to provide a description of their current "best practice" (see [7, 8]).

A. *How a statistical hypothesis is applied*. First of all, we have to admit that probability theory makes no predictions but only gives recommendations: *if the probability* (computed on the basis of

*LIF Marseille, CNRS & Univ. Aix–Marseille. On leave from IITP, RAS, Moscow. Supported in part by NAFIT ANR-08-EMER-008-01 grant. E-mail: `alexander.shen@lif.univ-mrs.fr`

the statistical hypothesis) *of an event A is much smaller than the probability of an event B, then the possibility of the event B must be taken into consideration to a greater extent than the possibility of the event A* (assuming the consequences are equally grave). For example, if the probability of A is smaller than the probability of being killed on the street by a meteorite, we usually ignore A completely (since we have to ignore event B anyway in our everyday life).

Borel [2] describes this principle as follows: "…Fewer than a million people live in Paris. Newspapers daily inform us about the strange events or accidents that happen to some of them. Our life would be impossible if we were afraid of all adventures we read about. So one can say that from a practical viewpoint we can ignore events with probability less that one millionth… Often trying to avoid something bad we are confronted with even worse… To avoid this we must know well the probabilities of different events" (Russian ed., pp. 159–160).

B. *How a statistical hypothesis is tested*. Here we cannot say naïvely that if we observe some event that has negligible probability according to our hypothesis, we reject this hypothesis. Indeed, this would mean that any 1000-bit sequence of the outcomes would make the fair coin assumption rejected (since this specific seqeunce has negligible probability 2^{-1000}).

Here algorithmic information theory comes into play: We reject the hypothesis if we observe a *simple* event that has negligible probability according to this hypothesis. For example, if coin tossing produces thousand tails, this event is simple and has negligible probability, so we don't believe the coin is fair. Both conditions ("simple" and "negligible probability") are important: the event "the first bit is a tail" is simple but has probability $1/2$, so it does not discredit the coin. On the other hand, every sequence of outcomes has negligible probability 2^{-1000}, but if it is not simple, its appearance does not discredits the fair coin assumption.

Often both parts of this scheme are united into a statement "events with small probabilities do not happen". For example, Borel writes: "One must not be afraid to use the word "certainty" to designate a probability that is sufficiently close to 1" ([3], Russian translation, p. 7). Sometimes this statement is called "Cournot principle". But we prefer to distinguish between these two stages, because for the hypothesis testing the existence of a simple description of an event with negligible probability is important, and for application of the hypothesis it seems unimportant. (We can expect, however, that events interesting to us have simple descriptions because of their interest.)

3 Simple events and events specified in advance

Unfortunately, this scheme remains not very precise: the Kolmogorov complexity of an object x (defined as the minimal length of the program that produces x) depends on the choice of programming language; we need also to fix some way to describe the events in question. Both choices lead only to an $O(1)$ change asymptotically; however, strictly speaking, due to this uncertainty we cannot say that one event has smaller complexity than the other one. (The word "negligible" is also not very precise.) On the other hand, the scheme described, while very vague, seems to be the best approximation to the current practice.

One of the possible ways to eliminate complexity in this picture is to say that a hypothesis is discredited if we observe a very unprobable event *that was specified in advance* (before the experiment). Here we come to the following question. Imagine that you make some experiment

and get a sequence of thousand bits that looks random at first. Then somebody comes and says "Look, if we consider every third bit in this sequence, the zeros and ones alternate". Will you still believe in the fair coin hypothesis? Probably not, even if you haven't thought about this event before looking at the sequence: the event is so simple that one *could* think about it. In fact, one may consider the union of all simple events that have small probability, and it still has small probability (if the bound for the complexity of a simple event is small compared to the number of coin tossing involved, which is a reasonable condition anyway). And this union can be considered as specified before the experiment (e.g., in this paper).

On the other hand, if the sequence repeats some other sequence observed earlier, we probably won't believe it is obtained by coin tossing even if this earlier sequence had high complexity. One may explain this opinion saying the the entire sequence of observations is simple since it contains repetitions; however, the first observation may be not covered by any probabilistic assumption. This could be taked into account by considering the *conditional* complexity of the event (with respect to all information available before the experiment).

The conclusion: we may remove one problematic requirement (being "simple" in a not well specified sense) and replace it by another problematic one (being specified before the observation).

4 Frequency approach

The most natural and common explanation of the notion of probability says that probability is the limit value of frequencies observed when the number of repetitions tends to infinity. (This approach was advocated as the only possible basis for probability theory by Richard von Mises.)

However, we cannot observe infinite sequences, so the actual application of this definition should somehow deal with finite number of repetitions. And for finite number of repetitions our claim is not so strong: we do not guarantee that frequency of tails for a fair coin is exactly $1/2$; we say only that it is highly improbable that it deviates significantly from $1/2$. Since the words "highly improbably" need to be interpreted, this leads to some kind of logical circle that makes the frequency approach much less convincing; to get out of this logical circle we need some version of Cournot principle.

Technically, the frequency approach can be related to the principles explained above. Indeed, the event "the number of tails in a $1\,000\,000$ coin tossings deviates from $500\,000$ more than by $100\,000$" has a simple description and very small probability, so we reject the fair coin assumption if such an event happens (and ignore the dangers related to this event if we accept the fair coin assumption). In this way the belief that frequency should be close to probability (if the statistical hypothesis is chosen correctly) can be treated as the consequence of the principles explained above.

5 Dynamical and statistical laws

We have described how the probability theory is usually applied. But the fundamental question remains: well, probability theory describes (to some extent) the behavior of a symmetric coin or dice and turns out to be practically useful in many cases. But is it a new law of nature or some

consequence of the known dynamical laws of classical mechanics? Can we somehow “prove” that a symmetric dice indeed has the same probabilities for all faces (if the starting point is high enough and initial linear and rotation speeds are high enough)?

Since it is not clear what kind of “proof” we would like to have, let us put the question in a more practical way. Assume that we have a dice that is not symmetric and we know exactly the position of its center of gravity. Can we use the laws of mechanics to find the probabilities of different outcomes?

It seems that this is possible, at least in principle. The laws of mecahnics determine the behavior of a dice (and therefore the outcome) if we know the initial point in the phase space (initial position and velocity) precisely. The phase space, therefore, is splitted into six parts that correspond to six outcomes. In this sense there is no uncertainty or probabilities up to now. But these six parts are well mixed since very small modifications affect the result, so if we consider a small (but not very small) part of the phase space around the initial conditions and any probability distribution on this part whose density does not change drastically, the measures of the six parts will follow the same proportion.

The last sentence can be transformed into a rigorous mathematical statement if we introduce specific assumptions about the size of the starting region in the phase space and variations of the density of the probability distribution on it. It then can be proved. Probably it is a rather difficult mathematical problem not solved yet, but at least theoretically the laws of mechanics allow us to compute the probabilities of different outcomes for a non-symmetic dice.

6 Are “real” sequences complex?

The argument in the preceding section would not convince a philosophically minded person. Well, we can (in principle) compute some numbers that can be interpreted as probabilities of the outcomes for a dice, and we do not need to fix the distribution on the initial conditions, it is enough to assume that this distribution is smooth enough. But still we speak about probability distributions that are somehow externally imposed in addition to dynamical laws.

Essentially the same question can be reformulated as follows. Make 10^6 coin tosses and try to compress the resulting sequence of zeros and ones by a standard compression program, say, gzip. (Technically, you need first to convert bit sequence into a byte sequence.) Repeat this experiment (coin tossing plus gzipping) as many times as you want, and this will never give you more that 1% compression. (Such a compression is possible for less than 2^{-10000}-fraction of all sequences.) This statement deserves to be called a law of nature: it can be checked experimentally in the same way as other laws are. So the question is: does this law of nature follows from dynamical laws we know?

To see where the problem is, it is convenient to simplify the situation. Imagine for a while that we have discrete time, phase space is $[0,1)$ and the dynamical law is

$$x \mapsto T(x) = \textbf{ if } 2x < 1 \textbf{ then } 2x \textbf{ else } 2x - 1.$$

So we get a sequence of states $x_0, x_1 = T(x_0), x_2 = T(x_1), \ldots$; at each step we observe where the current state is — writing 0 if x_n is in $[0,1/2)$ and 1 if x_n is in $[1/2,1)$.

This tranformation T has the mixing property we spoke about; however, looking at it more closely, we see that a sequence of bits obtained is just the binary representation of the initial condition. So our process just reveals the initial condition bit by bit, and any statement about the resulting bit sequence (e.g., its incompressibility) is just a statement about the initial condition.

So what? Do we need to add to the dynamical laws just one more methaphysical law saying that world was created at the random (=incompressible) state? Indeed, algorithmic transformations (including dynamical laws) cannot increase significantly the Kolmogorov complexity of the state, so if objects of high complexity exist in the (otherwise deterministic, as we assume for now) real world now, they should be there at the very beginning. (Note that it is difficult to explain the randomness observed saying that we just observe the world at random time or in a random place: the number of bits needed to encode the time and place in the world is not enough to explain an incompressible string of length, say 10^6, if we use currently popular estimates for the size and age of the world: the logarithms of the ratios of the maximal and minimal lengths (or time intervals) that exist in nature are negligible compared to 10^6 and therefore the position in space-time cannot determine a string of this complexity.

Should we conclude then that instead of playing the dice (as Einstein could put it), God provided concentrated randomness while creating the world?

7 Randomness as ignorance: Blum – Micali – Yao

This discussion becomes too philosophical to continue it seriously. However, there is an important mathematical result that could influence the opinion of the philosophers discussing the notions of probability and randomness. (Unfortunately, knowledge does not penetrate too fast, and I haven't yet seen this argument in traditional debates about the meaning of probability.)

This result is the existence of pseudorandom number generators (as defined by Yao, Blum and Micali; they are standard tools in computational cryptography, see, e.g., Goldreich textbook [4]). The existence is proved modulo some complexity assumtions (the existence of one-way functions) that are widely believed though not proven yet.

Let us explain what a pseudorandom number generator (in Yao – Blum – Micali) sense is. Here we use rather vague terms and oversimplify the matter, but there is a rigorious mathematics behind. So imagine a simple and fast algorithmic procedure that gets a "seed", which is a binary string of moderate size, say, 1 000 bits, and produces a very long sequence of bits out of it, say, of length 10^{10}. By necessity the output string has small complexity compared to its length (complexity is bounded by the seed size plus the length of the processing program, which we assume to be rather short). However, it may happen that the output sequences will be "indistinguishable" from truly random sequences of length 10^{10}, and in this case the transformation procedure is called pseudorandom number generator.

It sounds as a contradiction: as we have said, output sequences have small Kolmogorov complexity, and this property distinguishes them from most of the sequences of length 10^{10}. So how they can be indistinguishable? The explanation is that the difference becomes obvious only when we know the seed used for producing the sequence, but there is no way to find out this seed looking at the sequence itself. The formal statement is quite technical, but its idea is simple. Consider any

simple test that looks at 10^{10}-bit string and says ‘yes’ or ‘no’ (by whatever reason; any simple and fast program could be a test). Then consider two ratios: (1) the fraction of bit strings of length 10^{10} that pass the test (among all bit strings of this length); (2) the fraction of seeds that lead to a 10^{10}-bit string that passes the test (among all seeds). The pseudorandom number generator property guarantees that these two numbers are very close.

This implies that if some test rejects most of the pseudorandom strings (produced by the generator), then it would also reject most of the strings of the same length, so there is no way to find out whether somebody gives us random or pseudorandom strings.

In a more vague language, this example shows us that randomness may be in the eye of the beholder, i.e., the randomness of an observed sequence could be the consequence of our limited computational abilities which prevent us from discovering non-randomness. (However, if somebody shows us the seed, our eyes are immediately opened and we see that the sequence has very small complexity.)

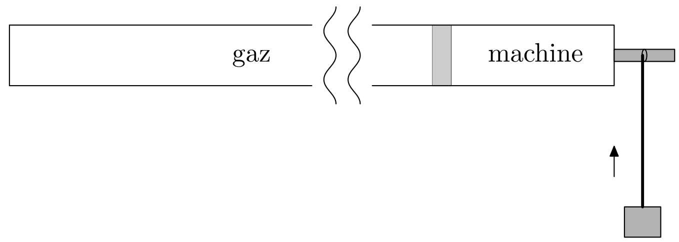
In particular, trying gzip-compression on pseudorandom sequences, we rarely would find them compressible (since gzip-compressibility is a simple test that fails for most sequences of length 10^{10}, it should also fail for most pseudorandom sequences).

So we should not exclude the possibility that the world is governed by simple dynamical laws and its initial state can be also described by several thousands of bits. In this case “true” randomness does not exist in the world, and every sequence of 10^6 coin tossing that happened or will happen in the foreseeable future produces a string that has Kolmogorov complexity much smaller than its length. However, a computationally limited observer (like ourselves) would never discover this fact.

8 Digression: thermodynamics

The connection between statistical and dynamical laws was discussed a lot in the context of thermodynamics while discussing the second law. However, one should be very careful with exact definition and statements. For example, it is often said that the Second Law of thermodynamics cannot be derived from dynamical laws because they are time-reversible while the second law is not. On the other hand, it is often said that the second law has many equivalent formulations, and one of them claims that the perpetual motion machine of the second kind is impossible, i.e., no device can operate on a cycle to receive heat from a single reservoir and produce a net amount of work.

However, as Nikita Markaryan explained (personal communication), in this formulation the second law of thermodynamics *is* a consequence of dynamic laws. Here is a sketch of this argument. Imagine a perpetual motion machine of a second kind exists. Assume this machine is attached to a long cylinder that contains warm gaz. Fluctuations of gaz pressure provide a heat exchange between gaz and machine. On the other side machine has rotating spindle and a rope to lift some weight (due to rotation).



When the machine works, the gaz temperature (energy) goes down and the weight goes up. This is not enough to call the machine a perpetual motion machine of the second kind (indeed, it can contain some amount of cold substance to cool the gaz and some spring to lift the weight). So we assume that the rotation angle (and height change) can be made arbitrarily large by increasing the amount of the gaz and the length of the cylinder. We also need to specify the initial conditions of the gaz; here the natural requirement is that the machine works (as described) for most initial conditions (according to the natural probability distribibution in the gaz phase space).

Why is such a machine impossible? The phase space of the entire system can be considered as a product of two components: the phase space of the machine itself and the phase space of the gaz. The components interact, and the total energy is constant. Since the machine itself has some fixed number of components, the dimension of its component (or the number of degrees of freedom in the machine) is negligible compared to the dimension of the gaz component (resp. the number of degrees of freedom in the gaz). The phase space of the gaz is splitted into layers corresponding to different level of energy; the higher the energy is, the more volume is used, and this dependence overweights the similar dependence for the machine since the gaz has much more degrees of freedom. Since the transformation of the phase space of the entire system is measure-preserving, it is impossible that a trajectory started from a large set with high probability ends in a small set: the probability of this event does not exceed the ratio of a measures of destination and source sets in the phase space.

This argument is quite informal and ignores many important points. For example, the measure on the phase space of the entire system is not exactly a product of measures on the gaz and machine coordinates; the source set of the trajectory can have small measure if the initial state of the machine is fixed with very high precision, etc. (The latter case does not contradicts the laws of thermodynamics: if the machine use a fixed amount of cooling substance of very low temperature, the amount of work produced can be very large.) But at least these informal arguments make plausible that dynamic laws make imposiible the perpetual motion machine of the second kind (if the latter is defined properly).

9 Digression: quantum mechanics

Another physics topic often discussed is quantum mechanics as a source of randomness. There were many philosophical debate around quantum mechanics; however, it seems that the relation between quantum mechanical models and observations resembles the situation with probability theory and statistical mechanics; in quantum mechanics the model assigns *amplitudes* (instead of probabilities) to different outcomes (or events). The amplitudes are complex numbers and “quan-

tum Cournot principle" says that if the (absolute value) of the amplitude of event A is smaller than for event B, then the possibility of the event B must be taken into consideration to a greater extent than the possibility of the event A (assuming the consequences are equally grave). Again this implies that we can (practically) ignore events with very small amplitudes.

The interpretation of the square of amplitude as probability can be then derived is the same way as in the case of the frequency approach. If a system is made of N independent identical systems with two outcomes 0 and 1 and the outcome 1 has amplutude z in each system, then for the entire system the amplitude of the event "the number of 1's among the outcomes deviates significantly from $N|z|^2$" is very small (it is just the classical law of large numbers in disguise).

One can then try to analyze measurement devices from the quantum mechanical viewpoint and to "prove" (using the same quantum Cournot principle) that the frequency of some outcome of measurement is close to the square of the length of the projection of the initial state to corresponding subspace outside some event of small amplitude, etc.

10 Acknowledgements

The material covered is not original: the topic was discussed for a long time in books and papers to numerous to mention, both the classics of the field (such as books and papers by Laplace, Borel, and Kolmogorov) and more expository writings, such as books written by Polya and Renyi. However, some remarks I haven't seen before and bear responsibility for all the errors and misunderstandings.

More information about the history of algorithmic information theory (Kolmogorov complexity, algorithmic randomness) can be found in [1].

I am indepted to many colleagues, including Vladimir Fock, Peter Gacs, Alexey Kitaev, Denis Kosygin, Leonid Levin, Leonid Levitov, Yury Makhlin, Nikita Markaryan (who explained to me the proof of the impossibility of the perpetuum mobile of the second kind), Vladimir A. Uspensky, Vladimir Vovk (who explained to me how the complexity assumption can be replaced by the requirement that the event should be specified before the experiment), Michael Vyalyi, Alexander Zvonkin, and all the participants of Kolmogorov seminar for many useful discussions.

I thank the organizers of LIX Colloquium "Reachability Problems'09" for the opportunity to present this material at the colloquium.

References

[1] Bienvenu, L., Shen, A.: Algorithmic information theory and martingales, preprint. Available as `arXiv:0906.2614`.

[2] Borel, E.: Le hazard, Alcan, Paris (1914). [Russian translation: Gosizdat, Moscow – Petrograd (1923)]

[3] Borel, E.: Probabilité et certitude, Presse Univ. de France, Paris (1950). [English translation: Probability and certainty, Walker publishers (1963). Russian translation: Fizmatgiz, Moscow (1961).]

[4] Goldreich, O.: Foundations of Cryptography. Volume 1. Basic Tools. Cambridge University Press (2007)

[5] Polya, G.: Mathematics and Plausible Reasoning, vols. 1,2. Princeton University Press (1990) [reprint]

[6] Renyi, A.: Letters on probability, Wayne State University Press (1972)

[7] Uspensky, V., Semenov, A., Shen, A.: Can an individual sequence of zeros and ones be random? Russian Mathematical Surveys, 45(1), 121–189 (1990)

[8] Shen, A.: On the logical basis of application of probability theory, In: Proc. Workshop on semiotic aspects of the formalization of intellectual activity, Telavi (Georgia, then USSR), 1983, pp. 144–146. VINITI, Moscow (1983)